

# A NOTE ON THE PROOF OF HÖLDER CONTINUITY TO WEAK SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. By borrowing ideas from the parabolic theory, we use a combination of De Giorgi's and Moser's methods to give some remarks on the proof of Hölder continuity of weak solutions of elliptic equations.

## 1. INTRODUCTION

We will present some observations on the proof of Hölder continuity of weak solutions to equations of type

$$\nabla \cdot \mathcal{A}(x, u, \nabla u) = 0. \quad (1.1)$$

This kind of elliptic equations are, of course, well-studied and there are many beautiful arguments for the Hölder regularity of their solutions. As it is well known, the problem was first solved independently by Ennio De Giorgi [2] and John Nash [13]. After Jürgen Moser used his iteration method for proving the supremum estimate, methods based on Harnack's inequalities were found as well [12], [14], [10], [11], [9].

Although the elliptic case is very well understood nowadays, the parabolic case seems to be more involved. In particular, there seems to be only one method for proving the continuity result for parabolic equations [3]. Consequently, a lot of research has been done for understanding the parabolic theory.

Using the ideas developed for the parabolic equations, we will give some remarks on the proof of Hölder continuity in the elliptic case. More precisely, we combine the De Giorgi method, in a form used in the parabolic setting, with Moser's iteration and a crossover lemma to give a proof for the regularity theorem.

The argument is formulated for a general Borel measure which is assumed to satisfy the doubling condition and to support a weak Poincaré inequality. These together are known to imply a Sobolev inequality which is the crucial tool we use. Regularity arguments for elliptic equations in the weighted case have been studied, for instance, by Fabes,

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Kenig and Serapioni in [4]. For further aspects of the theory see the classical book by Ladyzhenskaya and Ural'tseva [7].

## 2. PRELIMINARIES

Let  $\mu$  be a Borel measure and  $\Omega$  an open set in  $\mathbb{R}^d$ . The Sobolev space  $H^{1,p}(\Omega)$  is defined to be the completion of  $C^\infty(\Omega)$  with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \left( \int_{\Omega} |u|^p + |\nabla u|^p d\mu \right)^{1/p}.$$

A function  $u$  belongs to the local Sobolev space  $H_{loc}^{1,p}(\Omega)$  if it belongs to  $H^{1,p}(\Omega')$  for every  $\Omega' \Subset \Omega$ . Moreover, the Sobolev space with zero boundary values is defined as the completion of  $C_0^\infty(\Omega)$  with respect to the Sobolev norm. For more properties of Sobolev spaces, see e.g. [5] or [1].

Assume that  $\mathcal{A} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function such that  $\mathcal{A}(\cdot, \zeta, \xi)$  is measurable for every  $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^d$  and  $\mathcal{A}(x, \cdot, \cdot)$  is continuous for all  $x \in \Omega$ . Suppose also that for some  $\mathcal{A}_0 \geq 0$  and  $\mathcal{C}_0 > 0$  we have

$$|\mathcal{A}(x, \zeta, \xi)| \leq \mathcal{A}_0 |\xi|^{p-1}$$

and

$$\mathcal{A}(x, \zeta, \xi) \cdot \xi \geq \mathcal{C}_0 |\xi|^p.$$

A weak solution for equation (1.1) is defined as follows.

**Definition 2.1.** A function  $u \in H_{loc}^{1,p}(\Omega)$  is a weak solution of equation (1.1) in  $\Omega$  if it satisfies the integral equality

$$\int_{\Omega} \mathcal{A}(x, u, \nabla u) \cdot \nabla \phi d\mu = 0 \quad (2.2)$$

for all  $\phi \in C_0^\infty(\Omega)$ . If the equality in this definition is replaced by  $\geq$  ( $\leq$ ) and the inequality holds for every nonnegative  $\phi \in C_0^\infty(\Omega)$  we say that the function is a supersolution (subsolution).

The measure  $\mu$  is said to be doubling if there is a universal constant  $D_0 \geq 1$  such that

$$\mu(B(x, 2r)) \leq D_0 \mu(B(x, r))$$

for all  $B(x, 2r) \subset \Omega$ . Here  $B(x, r)$  denotes the standard open ball in  $\mathbb{R}^d$

$$B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}.$$

We will also use the notation

$$B(r) := B(0, r).$$

The dimension related to the doubling measure is defined by  $d_\mu := \log_2 D_0$ . Note that in the case of the Lebesgue measure  $d_{\mathcal{L}} = d$ . The

measure is said to support a weak  $(1, p)$ -Poincaré inequality if there exist constants  $P_0 > 0$  and  $\tau \geq 1$  such that

$$\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq P_0 r \left( \int_{B(x,\tau r)} |\nabla u|^p d\mu \right)^{1/p} \quad (2.3)$$

for every  $u \in H^{1,p}(\Omega)$  and  $B(x, \tau r) \subset \Omega$ . Here we used the notation

$$u_{B(x,r)} = \int_{B(x,r)} u d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u d\mu.$$

The word weak refers to the constant  $\tau \geq 1$ . If the inequality (2.3) is true for  $\tau = 1$  we say that the measure supports a  $(1, p)$ -Poincaré inequality.

It is known that the weak  $(1, p)$ -Poincaré inequality and the doubling condition imply a Sobolev embedding.

**Theorem 2.4.** *Suppose  $u \in H_0^{1,p}(B(x, r))$ . Then there is a constant  $C > 0$  such that*

$$\left( \int_{B(x,r)} |u|^\kappa d\mu \right)^{1/\kappa} \leq Cr \left( \int_{B(x,r)} |\nabla u|^p d\mu \right)^{1/p}$$

where

$$\kappa = \begin{cases} \frac{d_\mu p}{d_\mu - p}, & \text{for } 1 < p < d_\mu \\ 2p, & \text{otherwise.} \end{cases}$$

*Proof.* See for example [6]. □

We will also need the following lemma.

**Lemma 2.5.** *Let  $\{Y_n\}, n = 0, 1, 2, \dots$ , be a sequence of positive numbers, satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha}$$

*where  $C, b > 1$  and  $\alpha > 0$ . Then  $\{Y_n\}$  converges to zero as  $n \rightarrow \infty$ , provided*

$$Y_0 \leq C^{-1/\alpha} b^{1-\alpha^2}.$$

*Proof.* For the proof we refer to [3]. □

Our main theorem is the following well-known regularity result. The observations we make lie in the proof of the claim. More precisely, to deduce the claim we use a combination of De Giorgi's method and Moser's iteration scheme together with Chebyshev's inequality.

**Theorem 2.6.** *Suppose  $\mu$  is a doubling measure which supports a weak  $(1, p)$ -Poincaré inequality. Let  $u \in H_{loc}^{1,p}(\Omega)$  be a weak solution of equation (1.1). Then  $u$  is locally Hölder continuous.*

We will prove the Hölder continuity of the solution in a neighborhood of an arbitrary point. Since the equation is translation invariant, for simplicity of notation, we can assume this point to be the origin.

## 3. ESTIMATES FOR WEAK SOLUTIONS

Let us start by stating some classical lemmata.

**Lemma 3.1** (Caccioppoli). *Let  $u \geq 0$  be a weak subsolution for equation (1.1) in  $\Omega$ . Then there exists a constant  $C = C(p, \mathcal{A}_0, \mathcal{C}_0) > 0$  such that for every  $k \geq 0$  and  $\varphi \in C_0^\infty(\Omega)$  we have*

$$\int_{\Omega} |\nabla(u - k)_+|^p \varphi^p d\mu \leq C \int_{\Omega} (u - k)_+^p |\nabla \varphi|^p d\mu.$$

*Proof.* The result follows by choosing the test function  $\phi = (u - k)_+ \varphi^p$  in the definition of a weak solution. For details see [8].  $\square$

**Lemma 3.2** (Crossover). *Let  $u \geq 0$  be a weak supersolution for equation (1.1) in  $\Omega$  and let  $B(r) \Subset \Omega$ . Then there exist constants  $C$  and  $\delta > 0$  such that*

$$\left( \int_{B(r)} u^{-\delta} d\mu \right)^{1/\delta} \leq C \left( \int_{B(r)} u^{\delta} d\mu \right)^{-1/\delta}.$$

*Proof.* For the proof we refer to [5].  $\square$

**Lemma 3.3.** *Let  $u \geq 0$  be a weak solution of equation (1.1) in  $\Omega$  and let  $B(r) \Subset \Omega$ . Then for every  $\delta > 0$  there exists a constant  $C > 0$  such that*

$$\operatorname{ess\,sup}_{B(r/2)} u \leq C \left( \int_{B(r)} u^{\delta} d\mu \right)^{1/\delta}.$$

*Proof.* The result follows by standard iteration techniques, see [5].  $\square$

## 4. HÖLDER CONTINUITY

Let  $r > 0$  and denote

$$r_n := \frac{r}{2} + \frac{r}{2^{n+1}}, \quad B_n := B(r_n)$$

and

$$A_n := \{x \in B_n : u(x) > k_n\}$$

where

$$k_n := \operatorname{ess\,sup}_{B(r)} u - \frac{\operatorname{ess\,osc}_{B(r/2)} u}{2^{\lambda+1}} - \frac{\operatorname{ess\,osc}_{B(r/2)} u}{2^{\lambda+n+1}}$$

for  $n = 0, 1, 2, \dots$

**Lemma 4.1.** *Let  $u$  be a weak subsolution of equation (1.1) in  $B(r)$ . Then there exists a constant  $C > 0$  such that*

$$\frac{\mu(A_{n+1})}{\mu(B_{n+1})} \leq C 4^{n\kappa(1+1/p)} \left( \frac{\mu(A_n)}{\mu(B_n)} \right)^{\kappa/p}.$$

*Proof.* Choose the cut off function  $\varphi_n \in C_0^\infty(B_n)$  such that  $\varphi_n = 1$  in  $B_{n+1}$  and

$$|\nabla \varphi_n| \leq \frac{C2^n}{r}, \quad n = 1, 2, \dots$$

Using the doubling property of the measure together with Sobolev's inequality (Theorem 2.4) and the Caccioppoli inequality (Lemma 3.1) gives

$$\begin{aligned} & \int_{B_{n+1}} (u - k_n)_+^\kappa d\mu \\ & \leq \int_{B_{n+1}} (u - k_n)_+^\kappa \varphi_n^\kappa d\mu \leq \frac{\mu(B_n)}{\mu(B_{n+1})} \int_{B_n} (u - k_n)_+^\kappa \varphi_n^\kappa d\mu \\ & \leq Cr^\kappa \left( \int_{B_n} |\nabla (u - k_n)_+ \varphi_n|^p d\mu \right)^{\kappa/p} \\ & \leq Cr^\kappa \left( \int_{B_n} |\nabla (u - k_n)_+|^p \varphi_n^p + (u - k_n)_+^p |\nabla \varphi_n|^p d\mu \right)^{\kappa/p} \\ & \leq Cr^\kappa \left( \int_{B_n} (u - k_n)_+^p |\nabla \varphi_n|^p d\mu \right)^{\kappa/p} \\ & \leq C2^{n\kappa/p} \left( \int_{B_n} (u - k_n)_+^p d\mu \right)^{\kappa/p} \\ & \leq C2^{n\kappa/p} \left( \frac{\text{ess osc}_{B(r/2)} u}{2^\lambda} \right)^\kappa \left( \frac{\mu(A_n)}{\mu(B_n)} \right)^{\kappa/p}. \end{aligned}$$

On the other hand,

$$\int_{B_{n+1}} (u - k_n)_+^\kappa d\mu \geq \frac{\mu(A_{n+1})}{\mu(B_{n+1})} \left( \frac{\text{ess osc}_{B(r/2)} u}{2^{\lambda+n+2}} \right)^\kappa.$$

These together give

$$\frac{\mu(A_{n+1})}{\mu(B_{n+1})} \leq C4^{n\kappa(1+1/p)} \left( \frac{\mu(A_n)}{\mu(B_n)} \right)^{\kappa/p},$$

as required.  $\square$

Now by Lemma 2.5 we have  $\mu(A_n)/\mu(B_n) \rightarrow 0$  as  $n \rightarrow \infty$ , provided

$$\frac{\mu(A_0)}{\mu(B_0)} \leq C^{-1/(\kappa/p-1)} 4^{\kappa(1-(1-\kappa/p)^2)}. \quad (4.2)$$

Next we turn to prove that this will, indeed, be satisfied for some suitably chosen  $\lambda > 0$ .

**Lemma 4.3.** *Let  $u$  be a weak solution of equation (1.1) in  $B(3r)$ . Then there exists a constant  $\lambda_0 := \lambda > 0$  such that (4.2) holds. Recall that  $A_0$  depends on  $\lambda$ .*

*Proof.* Now by Chebyshev's inequality we have

$$\begin{aligned} \frac{\mu(A_0)}{\mu(B_0)} &= \mu(\{x \in B(r) : u > \operatorname{ess\,sup}_{B(r)} u - \frac{\operatorname{ess\,osc}_{B(r/2)} u}{2^\lambda}\}) / \mu(B_0) \\ &= \mu(\{x \in B(r) : \frac{\operatorname{ess\,osc}_{B(r/2)} u}{2^\lambda} > \operatorname{ess\,sup}_{B(r)} u - u\}) / \mu(B_0) \\ &\leq \left( \frac{\operatorname{ess\,osc}_{B(r/2)} u}{2^\lambda} \right)^\delta \int_{B(r)} \left( \frac{1}{\operatorname{ess\,sup}_{B(r)} u - u} \right)^\delta d\mu \end{aligned}$$

where  $\delta > 0$  is to be determined shortly. Since

$$\operatorname{ess\,sup}_{B(r)} u - u \geq 0$$

is a weak solution of equation (1.1) in  $B(3r)$ , by the Crossover lemma (Lemma 3.2) we have

$$\int_{B(r)} \left( \frac{1}{\operatorname{ess\,sup}_{B(r)} u - u} \right)^\delta d\mu \leq C \left( \int_{B(r)} (\operatorname{ess\,sup}_{B(r)} u - u)^\delta d\mu \right)^{-1}$$

for all small enough  $\delta > 0$ . By Lemma 3.3 we obtain

$$\begin{aligned} \left( \int_{B(r)} (\operatorname{ess\,sup}_{B(r)} u - u)^\delta d\mu \right)^{1/\delta} &\geq \frac{1}{C} \operatorname{ess\,sup}_{B(r/2)} (\operatorname{ess\,sup}_{B(r)} u - u) \\ &= \frac{1}{C} \left( \operatorname{ess\,sup}_{B(r)} u - \operatorname{ess\,inf}_{B(r/2)} u \right) \\ &\geq \frac{1}{C} \operatorname{ess\,osc}_{B(r/2)} u. \end{aligned}$$

Consequently,

$$\frac{\mu(A_0)}{\mu(B_0)} \leq \frac{C}{2^{\delta\lambda}}.$$

Choosing  $\lambda$  large enough finishes the proof.  $\square$

Now the Hölder estimate follows from the previous result by standard measures. For the sake of completeness we recall the argument in the form of the following theorem.

**Theorem 4.4.** *Let  $u$  be a weak solution of equation (1.1) in  $B(3r)$  and let  $x, y \in B(r)$  and  $r > |x - y|/2$ . Then there exist constants  $C > 0$  and  $0 < \alpha < 1$  such that*

$$|u(x) - u(y)| \leq C \left( \frac{|x - y|}{r} \right)^\alpha \operatorname{ess\,sup}_{B(3r)} |u|.$$

*Proof.* Either

$$\operatorname{ess\,osc}_{B(r/2)} u \leq \frac{1}{2} \operatorname{ess\,osc}_{B(r)} u$$

or by using the previous Lemma together with Lemmas 4.1 and 2.5 we obtain

$$u \leq \operatorname{ess\,sup}_{B(r)} u - \frac{\operatorname{ess\,osc}_{B(r)} u}{2^{\lambda_0+2}} \quad \text{a.e in } B(r/2) \quad (4.5)$$

for some  $\lambda_0 > 0$ , which only depends on the data. Now by subtracting  $\operatorname{ess\,inf}_{B(r/2)} u$  from both sides of (4.5) we obtain

$$\operatorname{ess\,osc}_{B(r/2)} u \leq \left(1 - \frac{1}{2^{\lambda_0+2}}\right) \operatorname{ess\,osc}_{B(r)} u. \quad (4.6)$$

We conclude that in any case (4.6) is true.

Let  $\gamma = (1 - 1/2^{\lambda_0+2})$ ,  $0 < r < R$  and choose  $i$  such that

$$\frac{R}{2^{i+1}} \leq r \leq \frac{R}{2^i}. \quad (4.7)$$

Now this together with an iteration of (4.6) gives

$$\operatorname{ess\,osc}_{B(r)} u \leq \operatorname{ess\,osc}_{B(R/2^i)} u \leq \gamma^i \operatorname{ess\,osc}_{B(R)} u \leq C \left(\frac{r}{R}\right)^\alpha \operatorname{ess\,osc}_{B(R)} u$$

where

$$\alpha = -\frac{\log \gamma}{\log 2}.$$

Let now  $x, y \in B(r)$  and, further, let  $R = 2r > |x - y|$ . Now we have

$$\begin{aligned} |u(x) - u(y)| &\leq \operatorname{ess\,osc}_{B((x+y)/2, |x-y|)} u \\ &\leq C \left(\frac{|x-y|}{R}\right)^\alpha \operatorname{ess\,osc}_{B((x+y)/2, R)} u \\ &\leq C \left(\frac{|x-y|}{r}\right)^\alpha \operatorname{ess\,sup}_{B(3r)} |u|, \end{aligned}$$

as required.  $\square$

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